Utledning av Heisenbergs Usikkerhetsprinsipp

La $\hat{\Psi}(p_x) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \Psi(x) e^{-ip_x x/\hbar}\, dx$ vare Fourieromvending av $\Psi(x)$,

hvor $\Psi(x) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \hat{\Psi}(p_x) e^{ip_x x/\hbar}\, dp_x$ er revers omvending.

Trenger Plancherels teorem:

$$\int_{-\infty}^{\infty} |\hat{\Psi}(p_x)|^2 dp_x = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \Psi(x) e^{-ip_x x/\hbar} dx \right) \left(\int_{-\infty}^{\infty} \overline{\Psi}(s) e^{ip_x s/\hbar} ds \right) dp_x$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(x)\overline{\Psi}(s) e^{ip_x(s-x)/\hbar}\, dx\, ds\, dp_x$$

innfør $k := \frac{p_x}{\hbar} \Rightarrow dk = \frac{1}{\hbar} dp_x$, $k(\pm\infty) = \pm\infty$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(x)\overline{\Psi}(s) e^{ik(s-x)}\, dx\, ds\, dk$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(x)\overline{\Psi}(s)\, \delta(s-x)\, dx\, ds$$

$$= \int_{-\infty}^{\infty} \Psi(x)\overline{\Psi}(x)\, dx = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx \,, \quad \blacksquare$$

trenger også derivasjonsregel for fourieromvending:

$$\widehat{\Psi'}(p_x) = \mathcal{F}\{\Psi'(x)\} = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \Psi'(x) e^{-ip_x x/\hbar}\, dx$$

$$= (2\pi\hbar)^{-1/2} \left[\Psi(x) e^{-ip_x x/\hbar} \right]_{x=-\infty}^{x=\infty} + \frac{ip_x}{\hbar} (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \Psi(x) e^{-ip_x x/\hbar}\, dx$$

(krever $\Psi(\pm\infty) = 0$)

$$= \frac{ip_x}{\hbar} \hat{\Psi}(p_x) \,.$$

$=$

$$\Delta x = \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \qquad \text{(antar } x \text{ sentrert} \Rightarrow \langle x \rangle = 0\text{)}$$

$$= \sqrt{\langle x^2 \rangle}$$

$$= \sqrt{\int_{-\infty}^{\infty} x^2 |\Psi(x)|^2\, dx} \,.$$

Trenger for evig også: $|(f,g)|^2 \leq (f,f)(g,g)$,

$|a+b| \leq |a|+|b|$, $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

$=$ (Born fortolkning)

$$1 = |1| = \left|\int_{-\infty}^{\infty} |\Psi(x)|^2 dx\right|$$

$$= \left|\int_{-\infty}^{\infty} |\Psi(x)|^2 \left(\frac{d}{dx}x\right) dx\right|$$

$$= \left|\int_{-\infty}^{\infty} x \frac{d}{dx}|\Psi(x)|^2 dx\right|$$

$$= \left|\int_{-\infty}^{\infty} x\Psi'(x)\overline{\Psi}(x) + x\Psi(x)\overline{\Psi}'(x)\, dx\right|$$

$$\leq 2\int_{-\infty}^{\infty} |x||\Psi(x)||\Psi'(x)|\, dx$$

$$\leq 2\sqrt{\int_{-\infty}^{\infty} x^2|\Psi(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} |\Psi'(x)|^2 dx}$$

$$= 2\sqrt{\int_{-\infty}^{\infty} x^2|\Psi(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} |\widehat{\Psi'}(p_x)|^2 dp_x}$$

$$= 2\sqrt{\int_{-\infty}^{\infty} x^2|\Psi(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} \frac{p_x^2}{\hbar^2}|\hat{\Psi}(p_x)|^2 dp_x}$$

$$= \frac{2}{\hbar}\sqrt{\int_{-\infty}^{\infty} x^2|\Psi(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} p_x^2|\hat{\Psi}(p_x)|^2 dp_x} \qquad \Big| \cdot \frac{\hbar}{2}$$

$$\Rightarrow \sqrt{\int_{-\infty}^{\infty} x^2|\Psi(x)|^2 dx}\sqrt{\int_{-\infty}^{\infty} p_x^2|\hat{\Psi}(p_x)|^2 dp_x} = \boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}$$